Introducing a direct method to solve nonlinear Volterra and Fredholm integral equations using orthogonal triangular functions

E. Babolian [a,*] Z. Masouri [a,†]
S. Hatamzadeh-Varmazyar [b,‡]

[a] Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran
[b] Department of Electrical Engineering, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran

Abstract

An effective direct method to determine numerical solutions of specific nonlinear integral equations is described. The method is based on vector forms for representation of triangular functions and its operational matrix. This approach needs no integration and uses no projection method such as collocation and Galerkin methods, so all calculations can be easily implemented. Some numerical examples are provided to illustrate accuracy and computational efficiency of the method. Also, the obtained results are compared with those of the direct method using BPFs.

Keywords: Nonlinear integral equations; Direct method; Vector forms; Triangular functions; Operational matrix.

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*babolian@tmu.ac.ir
† *Corresponding author.* Zahra Masouri. Postal address: P.O.Box: 14665-981, Tehran, Iran. Tel, Fax No.: +98-21-88258048. E-mail address: nmasouri@yahoo.com.
‡s.hatamzadeh@yahoo.com.

1 Introduction

Many physical and engineering problems lead to linear or nonlinear integral equations [2, 8, 9, 10]. A special case of these equations is Hammerstein integral equation. In recent years, several numerical methods for solving these equations have been presented. Some authors use decomposition method [1, 12, 14]. In most of these methods, a set of basis functions and an appropriate projection method such as Galerkin, collocation, etc. have been applied [5, 7, 11, 13]. These methods often transform an integral equation to a linear or nonlinear system of algebraic equations which can be solved by direct or iterative methods. In general, generating this system needs calculation of a large number of integrations.

This paper considers specific cases of Volterra and Fredholm integral equations of the forms:

$$x(s) + \lambda \int_0^s k(s,t)F(x(t))dt = y(s), \qquad 0 \leqslant s < 1, \tag{1}$$

and

$$x(s) + \lambda \int_0^1 k(s,t)F(x(t))dt = y(s), \qquad 0 \leqslant s < 1, \tag{2}$$

where the function $F(x(t))$ is a polynomial of $x(t)$ with constant coefficients. For convenience, we put $F(x(t)) = [x(t)]^n$ with n positive integer. Note that the method presented in this article can be easily extended and applied to any nonlinear integral equations of the forms Eqs. (1) and (2). It is clear that for $n = 1$, Eqs. (1) and (2) are linear integral equations of the second kind. Also, without loss of generality, it is supposed that the interval of integration is $[0, 1]$, since any finite interval $[a, b]$ can be transformed to interval $[0, 1]$ by linear maps [7].

For solving these equations, this paper uses a new set of orthogonal functions, introduced by Deb et al. [6]. These functions have been applied for solving variational problems by Babolian et al. [4]. In this article, we use vector forms of triangular functions (TFs), operational matrix of integration, expansion of functions of one and two variables with respect to the TFs, and other TFs properties introuduced by Babolian et al. [3]. By using these representations a nonlinear integral equation can be easily reduced to a nonlinear system of algebraic equations. The generation of this system needs just sampling of functions, multiplication and addition of matrices and does not use any integration.

Finally, we apply the proposed method on some examples and compare the obtained results with those of the direct method using BPFs. These comparisons show accuracy and efficiency of the proposed method.

2 Review of triangular functions vector forms

Triangular functions have been introuduced by Deb et al. [6] and studied and used by Babolian et al. [3, 4]. In this section, we review TFs vector forms and their properties proposed by Babolian et al. [3].

2.1 Definition and expansion

Two m-sets of triangular functions (TFs) are defined over the interval $[0, T)$ as [6]

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise,} \end{cases}$$
$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = T/m$, and $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function. In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$, and $h = 1/m$.

Now, let $\mathbf{T}(t)$ be a $2m$-vector defined as

$$\mathbf{T}(t) = \begin{pmatrix} \mathbf{T1}(t) \\ \\ \mathbf{T2}(t) \end{pmatrix}, \quad 0 \leqslant t < 1, \tag{4}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are defined as follows:

$$\mathbf{T1}(t) = [T1_0(t), T1_1(t), ..., T1_{m-1}(t)]^T,$$
$$\mathbf{T2}(t) = [T2_0(t), T2_1(t), ..., T2_{m-1}(t)]^T, \tag{5}$$

in which $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

Now, the expansion of $f(t)$ with respect to TFs can be written as

$$\begin{aligned} f(t) &\simeq F1^T\mathbf{T1}(t) + F2^T\mathbf{T2}(t) \\ &= F^T\mathbf{T}(t), \end{aligned} \tag{6}$$

where $F1$ and $F2$ are TFs coefficients with $F1_i = f(ih)$ and $F2_i = f((i+1)h)$, for $i = 0, 1, \ldots, m-1$. Also, $2m$-vector F is defined as follows:

$$F = \begin{pmatrix} F1 \\ \\ F2 \end{pmatrix}. \tag{7}$$

Now, assume that $k(s,t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$k(s,t) \simeq \mathbf{T}^T(s)\ K\ \mathbf{T}(t), \tag{8}$$

where $\mathbf{T}(s)$ and $\mathbf{T}(t)$ are $2m_1$ and $2m_2$ dimensional triangular functions and K is a $2m_1 \times 2m_2$ TFs coefficient matrix. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} (K11)_{m\times m} & (K12)_{m\times m} \\ \\ (K21)_{m\times m} & (K22)_{m\times m} \end{pmatrix}, \tag{9}$$

where $K11$, $K12$, $K21$, and $K22$ can be computed by sampling the function $k(s,t)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, \ldots, m$. Therefore

$$\begin{aligned} (K11)_{i,j} &= k(s_i,t_j), \quad i = 0,1,\ldots,m-1, \quad j = 0,1,\ldots,m-1, \\ (K12)_{i,j} &= k(s_i,t_j), \quad i = 0,1,\ldots,m-1, \quad j = 1,2,\ldots,m, \\ (K21)_{i,j} &= k(s_i,t_j), \quad i = 1,2,\ldots,m, \quad j = 0,1,\ldots,m-1, \\ (K22)_{i,j} &= k(s_i,t_j), \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,m. \end{aligned} \tag{10}$$

2.2 Product properties

Let X be a $2m$-vector which can be written as $X^T = (X1^T \quad X2^T)$ such that $X1$ and $X2$ are m-vectors. Now, it can be concluded that

$$\mathbf{T}(t)\mathbf{T}^T(t)X \simeq \tilde{X}\mathbf{T}(t), \tag{11}$$

where $\tilde{X} = diag(X)$ is a $2m \times 2m$ diagonal matrix.

Now, let B be a $2m \times 2m$ matrix. So, it can be similarly concluded that

$$\mathbf{T}^T(t) B \mathbf{T}(t) \simeq \hat{B}^T \mathbf{T}(t), \tag{12}$$

in which $\hat{B}$ is a $2m$-vector with elements equal to the diagonal entries of matrix B.

Also,

$$\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)\ dt \simeq D, \tag{13}$$

where D is the following $2m \times 2m$ matrix:

$$D = \begin{pmatrix} \frac{h}{3} I_{m\times m} & \frac{h}{6} I_{m\times m} \\ \\ \frac{h}{6} I_{m\times m} & \frac{h}{3} I_{m\times m} \end{pmatrix}. \tag{14}$$

2.3 Operational matrix

Expressing $\int_0^s \mathbf{T}(\tau) d\tau$ in terms of $\mathbf{T}(s)$, we can write

$$\int_0^s \mathbf{T}(\tau) d\tau \simeq P\mathbf{T}(s), \tag{15}$$

where $P_{2m\times 2m}$, operational matrix of $\mathbf{T}(s)$, is

$$P = \begin{pmatrix} P1 & P2 \\ \\ P1 & P2 \end{pmatrix}, \tag{16}$$

where $P1$ and $P2$ are the TFs operational matrix of integration as follows [6]

$$P1 = \frac{h}{2} \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P2 = \frac{h}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{17}$$

Now, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned} \int_0^s f(\tau) d\tau &\simeq \int_0^s F^T \mathbf{T}(\tau) d\tau \\ &\simeq F^T P \mathbf{T}(s). \end{aligned} \tag{18}$$

3 solving nonlinear integral equations

In this section, using the results obtained above, an effective and very accurate direct method for solving nonlinear integral equations of the forms (1) and (2) is presented. Actually, the method will be described to solve both Volterra and Fredholm integral equations.

First, we require to prove the followig lemma.

Lemma 1. *Let $2m$-vectors X and X_n be TFs coefficients of $x(s)$ and $[x(s)]^n$ respectively. If*

$$X = (X1^T \quad X2^T)^T = (X1_0, X1_1, \ldots, X1_{m-1}, X2_0, X2_1, \ldots, X2_{m-1})^T \tag{19}$$

then

$$X_n = (X1_0^n, X1_1^n, \ldots, X1_{m-1}^n, X2_0^n, X2_1^n, \ldots, X2_{m-1}^n)^T \tag{20}$$

where $n \geqslant 1$, is a positive integer.

Proof. When $n = 1$, (20) follows at once from $[x(s)]^n = x(s)$. Suppose that (20) holds for n, we shall deduce it for $n + 1$.

Since $[x(s)]^{n+1} = x(s)[x(s)]^n$, from (6) and (11) follows

$$\begin{aligned}
[x(s)]^{n+1} &\simeq (X^T\mathbf{T}(s)).(X_n^T\mathbf{T}(s)) \\
&= X^T\mathbf{T}(s)\mathbf{T}^T(s)X_n \\
&\simeq X^T\tilde{X}_n\mathbf{T}(s).
\end{aligned} \tag{21}$$

Now, using (20) we obtain

$$X^T\tilde{X}_n = (X1_0^{n+1}, X1_1^{n+1}, \ldots, X1_{m-1}^{n+1}, X2_0^{n+1}, X2_1^{n+1}, \ldots, X2_{m-1}^{n+1})^T. \tag{22}$$

Therefore (20) holds for $n + 1$, and the lemma is established. □

So, the components of X_n can be computed in terms of components of unknown vector X.

3.1 Nonlinear Volterra integral equation

Consider the following nonlinear Volterra integral equation:

$$x(s) + \lambda \int_0^s k(s,t)[x(t)]^n dt = y(s), \qquad 0 \leqslant s < 1, \quad n \geqslant 1, \tag{23}$$

where the parameter λ and the functions $y(s)$ and $k(s,t)$ are known but $x(s)$ is not. Moreover, $k(s,t) \in \mathcal{L}^2([0,1) \times [0,1))$ and $y(s) \in \mathcal{L}^2([0,1))$.

Approximating the functions $x(s)$, $[x(s)]^n$, $y(s)$ and $k(s,t)$ with respect to TFs, (6) and (8) gives

$$\begin{aligned} x(s) &\simeq X^T\mathbf{T}(s) = \mathbf{T}^T(s)X, \\ [x(s)]^n &\simeq X_n^T\mathbf{T}(s) = \mathbf{T}^T(s)X_n, \\ y(s) &\simeq Y^T\mathbf{T}(s) = \mathbf{T}^T(s)Y, \\ k(s,t) &\simeq \mathbf{T}^T(s)K\mathbf{T}(t), \end{aligned} \tag{24}$$

where $2m$-vectors X, X_n, Y, and $2m \times 2m$ matrix K are TFs coefficients of $x(s)$, $[x(s)]^n$, $y(s)$, and $k(s,t)$, respectively.

For solving Eq. (23), we substitute (24) into (23). Therefore

$$Y^T\mathbf{T}(s) \simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\int_0^s \mathbf{T}(t)\mathbf{T}^T(t)X_n dt. \tag{25}$$

Using Eq. (11) and operational matrix P, in Eq. (15), follows

$$\begin{aligned} Y^T\mathbf{T}(s) &\simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\tilde{X}_n\int_0^s \mathbf{T}(t)dt \\ &\simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\tilde{X}_nP\mathbf{T}(s), \end{aligned} \tag{26}$$

in which $\lambda K\tilde{X}_nP$ is a $2m \times 2m$ matrix. Eq. (12) follows

$$\mathbf{T}^T(s)\lambda K\tilde{X}_nP\mathbf{T}(s) \simeq \hat{X}_n^T\mathbf{T}(s), \tag{27}$$

where $\hat{X}_n$ is a $2m$-vector with components equal to the diagonal entries of the matrix $\lambda K\tilde{X}_nP$.

Now, combining (26) and (27) and replacing $\simeq$ with $=$ gives

$$X + \hat{X}_n = Y. \tag{28}$$

Equation (28) is a nonlinear system of $2m$ algebraic equations for the $2m$ unknowns $X1_0, X1_1, \ldots, X1_{m-1}, X2_0, X2_1, \ldots, X2_{m-1}$, components of $X^T = (X1^T \quad X2^T)$, which can be obtained by an iterative method. Hence, an approximate solution $x(s) \simeq X^T\mathbf{T}(s)$, or $x(s) \simeq X1^T\mathbf{T1}(s) + X2^T\mathbf{T2}(s)$ can be computed for Eq. (23) without using any projection method.

3.2 Nonlinear Fredholm integral equation

Consider the following nonlinear Fredholm integral equation:

$$x(s) + \lambda \int_0^1 k(s,t)[x(t)]^n dt = y(s), \qquad 0 \leqslant s < 1, \quad n \geqslant 1, \tag{29}$$

where the parameter λ and the functions $y(s)$ and $k(s,t)$ are known and $x(s)$ is the unknown function to be determined. Moreover, $k(s,t) \in \mathcal{L}^2([0,1) \times [0,1))$ and $y(s) \in \mathcal{L}^2([0,1))$.

Similar to the direct method for Volterra integral equation, substituting Eqs. (24) into (29) follows

$$Y^T\mathbf{T}(s) \simeq X^T\mathbf{T}(s) + \lambda \mathbf{T}^T(s) K \int_0^1 \mathbf{T}(t)\mathbf{T}^T(t) X_n dt. \tag{30}$$

Using Eq. (13) follows

$$Y^T\mathbf{T}(s) \simeq X^T\mathbf{T}(s) + (\lambda K D X_n)^T \mathbf{T}(s). \tag{31}$$

Now, replacing $\simeq$ with $=$ gives

$$X + \lambda K D X_n = Y. \tag{32}$$

Equation (32) is a nonlinear system of algebraic equations. So, there is an approximate solution $x(s) \simeq X^T\mathbf{T}(s) = X1^T\mathbf{T1}(s) + X2^T\mathbf{T2}(s)$ for Eq. (29). Note that this approach does not use any projection method such as collocation, Galerkin, etc.

4 Numerical examples

The direct method presented in this article is applied to solve some examples. Also, the numerical results obtained here are compared with the exact solutions and the approximate solutions obtained by the method proposed in [2] which is implemented using the BPFs.

The computations associated with the examples have been performed using Matlab 7 on a personal computer having the Intel Pentium 4 2.5 GHz processor.

Table 1: Numerical results for example 1

s	Exact solution	Proposed method ($m = 4$)	Proposed method ($m = 8$)	BPFs method ($m = 8$)
0	0	0	0	0.062459
0.1	0.099833	0.099479	0.099870	0.062459
0.2	0.198669	0.198958	0.198569	0.186402
0.3	0.295520	0.295329	0.295330	0.307429
0.4	0.389418	0.388590	0.389397	0.423650
0.5	0.479426	0.481852	0.480026	0.533246
0.6	0.564642	0.563070	0.564655	0.533246
0.7	0.644218	0.644288	0.643792	0.634505
0.8	0.717356	0.716949	0.716834	0.725841
0.9	0.783327	0.781053	0.783210	0.805826

Example 1. Consider the following nonlinear Volterra integral equation [2, 14]

$$x(s) - \frac{1}{2}\int_0^s x^2(t)dt = \sin s + \frac{1}{8}\sin 2s - \frac{1}{4}s, \tag{33}$$

with the exact solution $x(s) = \sin s$. The numerical results are shown in Table 1.

Example 2. For the following nonlinear Volterra integral equation [14]

$$x(s) - \int_0^s s\, x^3(t)dt = e^s - \frac{1}{3}se^{3s} + \frac{s}{3}, \tag{34}$$

with the exact solution $x(s) = e^s$, Table 2 shows the numerical results.

Example 3. For the following nonlinear Fredholm integral equation [2, 14]

$$x(s) - \frac{1}{2}\int_0^1 t\, x^2(t)dt = s^2 - \frac{1}{12}, \tag{35}$$

with the exact solution $x(s) = s^2$, the numerical results are shown in Table 3.

Table 2: Numerical results for example 2

s	Exact solution	Proposed method ($m = 32$)	Proposed method ($m = 64$)	BPFs method ($m = 64$)
0	1	1	1	1.007833
0.1	1.105171	1.105267	1.105206	1.106890
0.2	1.221403	1.221589	1.221437	1.215682
0.3	1.349859	1.350140	1.349916	1.356193
0.4	1.491825	1.492237	1.491942	1.489490
0.5	1.648721	1.649408	1.648892	1.661654
0.6	1.822119	1.823938	1.822586	1.824987
0.7	2.013753	2.018624	2.014918	2.004406
0.8	2.225541	2.241868	2.229399	2.236313
0.9	2.459603	2.540585	2.477948	2.456052

Table 3: Numerical results for example 3

s	Exact solution	Proposed method ($m = 16$)	Proposed method ($m = 32$)	BPFs method ($m = 32$)
0	0	0.000651	0.000163	0.000217
0.1	0.010000	0.011588	0.010319	0.011936
0.2	0.040000	0.041276	0.040397	0.041233
0.3	0.090000	0.091276	0.090397	0.088108
0.4	0.160000	0.161588	0.160319	0.152561
0.5	0.250000	0.250651	0.250163	0.265842
0.6	0.360000	0.361588	0.360319	0.371311
0.7	0.490000	0.491276	0.490397	0.494358
0.8	0.640000	0.641276	0.640397	0.634983
0.9	0.810000	0.811588	0.810319	0.793186

Table 4: Numerical results for example 4

s	Exact solution	Proposed method ($m = 8$)	Proposed method ($m = 16$)	BPFs method ($m = 16$)
0	0	0	0	0.031228
0.1	0.099833	0.099771	0.099796	0.093563
0.2	0.198669	0.198352	0.198613	0.216902
0.3	0.295520	0.294978	0.295438	0.277425
0.4	0.389418	0.388893	0.389231	0.394996
0.5	0.479426	0.479358	0.479407	0.506412
0.6	0.564642	0.563807	0.564340	0.559265
0.7	0.644218	0.642755	0.643947	0.658227
0.8	0.717356	0.715601	0.717036	0.703950
0.9	0.783327	0.781774	0.782821	0.786989

Example 4. Consider the following nonlinear Fredholm integral equation

$$x(s) + \int_0^1 (s^2 - st)\ x^3(t)dt = y(s). \tag{36}$$

$y(s)$ is chosen such that the exact solution is $x(s) = \sin s$. Table 4 shows the numerical results.

The average times necessary to run the programs associated with the two methods are given in Table 5 which confirm that the proposed method is implemented and run very quickly in comparison with the BPFs method . The times are in seconds.

5 Error evaluation

The direct method based on TFs and its operational matrix, without applying any projection method, transforms a nonlinear Volterra or Fredholm integral equation to a set of algebraic equations. Its applicability and accuracy were checked on some examples. In these examples the approximate solutions are briefly compared with the exact and approximate solutions obtained by

Table 5: The average times necessary to run the methods (in seconds)

The methods	$m = 4$	$m = 8$	$m = 16$	$m = 32$	$m = 64$
Proposed method	0.12	0.13	0.17	0.30	0.59
BPFs method	0.30	0.45	1.12	4.40	27.5

the method proposed in [2]. It follows from the numerical results that the accuracy of the solutions obtained using the TFs is quite good in comparison with the BPFs. To show the convergence and stability of this approach, the mean-absolute errors at the points s in Tables 1-4 are computed for different values of m.

Consider the mean-absolute error as follows:

$$E_n^{(m)} = \frac{1}{n}\sum_{i=1}^{n} |x(s_i) - x_m(s_i)|, \tag{37}$$

where $x(s)$ is the exact solution and $x_m(s)$ is the approximate solution.

For example 1, the mean-absolute errors for ten points s in Table 1 are $8.4E-4$, for $m = 4$ and $2E-4$, for $m = 8$. These errors for example 2 at the points s in Table 2 are $1.1E-2$ and $2.4E-3$, for $m = 32$ and $m = 64$, respectively. The mean-absolute errors of the example 3 at the points s in Table 3 are $1.3E-3$ and $3.2E-4$, for $m = 16$ and $m = 32$, respectively. For example 4, the errors for the points s in Table 4, are $7.1E-4$, for $m = 8$ and $1.8E-4$, for $m = 16$. These errors show that increasing the number of TFs over $[0,1)$ decreases the error of the solution rapidly.

[14] proposes the decomposition method to solve the examples 1-3. It seems that the direct method is more accurate and practical than the decomposition method. On the other hand, the number of calculations of the direct method is lower.

The method presented in [2] uses the BPFs to obtain the numerical solutions of integral equations given in the examples 1 and 3. Comparing the results obtained here with the results presented in [2] shows that the current method is more accurate and the number of its calculations is lower.

The numerical results given in Tables 1-4 show the convergence and good stability of the current method. So, one can run the method with increasing m until the computed results have an appropriate accuracy.

6 Conclusion

This article introduced a numerical method to solve nonlinear Volterra and Fredholm integral equations. Using the Lemma presented in section 4, new vector forms of the TFs, and operational matrix of integration, this approach transforms an integral equation to a system of algebraic equations directly. This system can be solved by an iterative method.

The benefits of this method are low cost of setting up the equations without applying any projection method such as Galerkin and collocation methods, and using no integration to approximate the functions. This results in very quick implementation of the method. Moreover, the nonlinear system of algebraic equations is sparse and the numerical results have very good accuracy.

Finally, this method can be easily extended and applied to systems of nonlinear integral equations.

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